KERR METRIC FOR THE ELECTRON   N. Albers,   11/28/2008

The angular momentum singularity is elucidated by null geodesics in Cartesian coordinates.

Part I: Learning to crawl

The general Kerr metric may be written in axially symmetric coordinates [1] (transformed from Cartesian) as:

$$ds^2 = \left(1 - \frac{2m\rho}{\rho^2 + a^2\cos^2\hat{\theta}}\right) c^2\, d\hat{t}^2 \; - \; \frac{\rho^2 + a^2\cos^2\hat{\theta}}{\rho^2 + a^2 - 2m\rho}\, d\rho^2 \; - \; (\rho^2 + a^2\cos^2\hat{\theta})\, d\hat{\theta}^2 \; -$$

$$\left[(\rho^2 + a^2)\sin^2\hat{\theta} + \frac{2m\rho\, a^2 \sin^4\hat{\theta}}{\rho^2 + a^2\cos^2\hat{\theta}}\right] d\hat{\phi}^2 \; - \; \frac{4\, m\rho a\, \sin^2\hat{\theta}}{\rho^2 + a^2\cos^2\hat{\theta}}\, cdt\; d\hat{\phi} \quad . \qquad (1)$$

Since the Schwarzschild radius m of electron is in the range of E-57 meters, and the “angular momentum radius” a is about E-13 meters, we can in some instances drop terms in *m.* There is, however, a severe analytic difference here, in that ρ can go strictly to zero in the range for the original Cartesian r between 0 and a. In these cases assumptions must be reexamined, as one can see with the denominator in the first term. If the angular cosine can go to zero then the fraction blows up as $1/\rho$, as ρ nears $2m$. One might choose to be unconcerned about behavior at such small radii, but this occurs also near $r=a$. Let us examine the degeneracy in this radial transform.

Stepping back a bit, we must get familiar with the coordinate transforms which got us to this point. We defined ρ as the real part of the generalized radius, which was allowed a complex part: $\omega^2 = x^2 + y^2 + (z - ia)^2$. We define real and imaginary parts: $\omega \equiv \rho + i\sigma$ but will find a near range where the real part does not exist. Solving for this: $\rho^2 - \sigma^2 = r^2 - a^2$ where r is the original Cartesian radius,

and: $\sigma = \frac{-az}{\rho}$. Algebraically we can see the possibility of the sums producing negative ρ^2 so let us investigate the locus where: $\rho = 0$. A polar angle is newly defined: $\cos\hat{\theta} \equiv \frac{z}{\rho}$ so we can put this in for σ and get:

$$\rho^2 - a^2\cos^2\hat{\theta} = r^2 - a^2 \quad . \tag{2}$$

If we try to solve for zero ρ, $r^2 - a^2 + a^2\cos^2\hat{\theta} = 0$, or: $r^2 - a^2\sin^2\hat{\theta} = 0$. (3)

This seems reasonable and the confusion arises because both z and ρ go to zero. This is built into the definitions so that ρ is always at least as great as $|z|$. Leaving in terms with z rather than $\cos\hat{\theta}$, we can write the quadratic solution:

$$\rho^2 = \frac{1}{2}(r^2 - a^2) + \left[\frac{1}{4}(r^2 - a^2)^2 + a^2 z^2\right]^{1/2} \quad . \tag{4}$$

Here the plus sign has been chosen to get farfield consistency, and it gives a curious turnaround as r becomes less than a. Let the z-component be zero, then the square of ρ is still defined and is zero for $r<a$. For $r>a$, the two terms add and there is positive ρ. Thus the locus of zero ρ is $z=0$, $r \le a$.

There is a degeneracy of information, however, which actually has been transferred to the other variables. The coordinate transform by which the final coordinates were expressed defines azimuth angle ϕ as:

$$(\rho - ia)e^{i\bar{\Phi}}\sin\hat{\theta} \equiv x + iy \tag{5}$$

where ρ and $\hat{\theta}$ have already been defined. Let us ask about points on the disc of $z=0$ and $r \le a$. When $r=a$ the regime of $\rho=0$ begins, as we go inward, so we can write the above:

$$-ia\,e^{i\bar{\Phi}}\sin\hat{\theta} \equiv x + iy \quad . \tag{6}$$

Looking at the magnitude of each side, here the radius of the circle in $<x,y>$ is a. Thus the polar angle $\hat{\theta}$ is $\pi/2$. As we come "in" to lesser radii in $<x,y>$, however, *the polar angle folds upward by definition.* Thus it moves smoothly toward a definition:

$$\sin\hat{\theta} = 0 \qquad \text{where} \quad <x,y,z> = 0.$$

This illuminates our earlier confusion. One could sense that z/ρ neared unity, but this is only true near the center in *r*. The mapping into $\hat{\theta}$ can be understood if we take the magnitude of both sides when both z and ρ are zero. Thus in this locus,

$$\sin\hat{\theta}=r/a \quad . \tag{7}$$

There were other transforms on the way to this analysis which we will now list. First was the Eddington transform of the time variable to get degenerate form:

$$\bar{x}^0=x^0+2m\log\left|\frac{r}{2m}-1\right| \quad . \tag{8}$$

Then after the establishment of $\langle\rho,\hat{\theta},\bar{\phi}\rangle$, there are two other transforms in $\langle\bar{x}^{0,}\bar{\phi}\rangle$ to the final coordinates, $\langle\hat{x}^0,\rho,\hat{\theta},\hat{\phi}\rangle$, which will be dealt with as needed. They are introduced to eliminate differential cross-terms in the metric expression.

Part II: Learning to walk

Having an awareness of the coordinate behavior we may proceed to investigate *null geodesics,* or possible light-paths characterized by: $ds^2=0$. Our expressions are in transformed coordinates and my goal is to examine the field in the original Cartesian “external” variables. We shall deal with the transforms as we need to. The first step is to examine the locus $\rho=0$; as long as this is so, $d\rho=0$. Thus we need not consider the first spatial term in the metric. In fact, changes of radial measure r are mapped into changes of $\hat{\theta}$, as noted above. Differentiating both sides of the relevant constraint: $d(\sin\hat{\theta})=\frac{1}{a}dr=\cos\hat{\theta}\,d\hat{\theta}$. (9)

This is conveniently substituted to get terms in $\langle r, dr\rangle$ and the metric form in this instance now reads: $ds^2=c^2d\hat{t}^2-dr^2$, (10)

with a caveat: we are allowed to zero out the fraction multiplying the time differential only as long as there is non-zero $\cos\hat{\theta}$. We must approach the singular point at *r=a* with caution, though for any *r<a* (and *z=0*) the term is strictly unity. Later on we will investigate the limit approaching this point from the outside. To deal with the time coordinate, the Eddington transform implies:

$$d\bar{x}^0 = dx^0 + \frac{dr}{(r/2m-1)} \quad , \tag{11}$$

and this can be substituted into the metric expression. The final transforms involved adding different terms in ρ to the time variable, but as long as we remain in the locus of ρ=0, nothing will change in our present discussion. Now we may say:

$$ds^2 = \left[dx^0 + \frac{dr}{r/2m-1} \right]^2 - dr^2 \quad , \tag{12}$$

and to investigate null geodesics we set the LHS to zero, enabling us to write:

$$\pm dr = dx^0 + \frac{dr}{r/2m-1} \quad , \tag{13}$$

and then:

$$dx^0 = dr\left[\pm 1 - \frac{1}{r/2m-1}\right] \quad . \tag{14}$$

Observing that the second term is “usually” small, we choose the plus sign:

$$\frac{dr}{dx^0} = \frac{r/2m-1}{r/2m-2} \quad . \tag{15}$$

This shows dramatic changes only at very small radii, and answers our quest in the region *z=0, 0<r<a.*

Let us ask now about angular propagations in ϕ, which is to say differentials $d\hat{\phi}$. Near the plane of *z=0*, when *r<a*, the equations become simple if we look at changes only in that variable. Here:

$$ds^2 = (d\hat{x}^0)^2 - a^2 \sin^2\theta \, d\hat{\phi}^2 \quad , \tag{16}$$

and we may say:

$$ds^2 = (d\hat{x}^0)^2 - r^2 d\hat{\phi}^2 \quad . \tag{17}$$

As long as are looking at movements only in that coordinate, the original transform indicates that: $d\hat{\phi} = d\phi$, so after we realize that without a differential in r, the Eddington term is transparent, the problem is solved with an unchanged value:

$$r\, d\phi = d\hat{x}^0 \quad . \tag{18}$$

We thus get the same behavior as for radial differentials, with the Eddington term contributing as before.

Consider now differential movements in z for $r<a$ from the $<x,y>$-plane. We are accustomed to thinking of these as represented by a change in θ, but this is not the case for the transformed coordinates in this domain. Indeed a dz is mapped into $d\rho$. Looking at the solution for $\rho(r,z)$, we can see that if $R^2 \equiv a^2 - r^2$, then:

$$\rho^2 = -R^2/2 + \sqrt{R^4/4 + a^2 z^2} \quad , \tag{19}$$

and if z is small:

$$\rho^2 = R^2/2\left[-1 + \sqrt{1 + \frac{4a^2 z^2}{R^2}}\right] \approx \frac{a^2 z^2}{R^2} \quad . \tag{20}$$

This is a smooth continuation of the definition of $\cos\hat{\theta}$, so we can say for small deviations, $d\rho = \dfrac{dz}{\cos\hat{\theta}}$. The metric form is now: $ds^2 = (d\hat{x}^0)^2 - dz^2$, where the term comes from substituting into the $(d\rho)^2$ differential. Thus, like the other two Cartesian directions, this has the Schwarzschild dependence at $r \sim 2\mathrm{m}$. It seems there is isotropy here.

Part III: Ariadne's Thread

In the domain z=0, let us look first at behavior for radial changes for $r \geq a$. Here there is non-zero $d\rho$, though we may say that strictly: $\cos\hat{\theta} = 0$.

The metric form is:

$$ds^2 = \left(1-2\,m/\rho\right)(d\,\hat{x}^0)^2 - \frac{\rho^2}{\rho^2+a^2}d\,\rho^2 - \rho^2 d\,\hat{\theta}^2 - \ldots \quad , \tag{21}$$

and in this regime we have: $\rho^2+a^2 = r^2$ and may say: $\rho\,d\,\rho = r\ d\,r$.

These relations are substitued into the radial term to yield:

$$ds^2 = \left(1-2\frac{m}{\sqrt{r^2-a^2}}\right)(d\,\hat{x}^0)^2 - dr^2 \quad . \tag{22}$$

Other transformations do not confuse things, as before, and we can substitute with the Eddington transform:

$$ds^2 = \left(1-\frac{2\,m}{\sqrt{r^2-a^2}}\right)\left(dx^0 + \frac{dr}{r/2\,m-1}\right)^2 - dr^2 \quad . \tag{23}$$

We are not considering changes in polar angle, since outside of *r=a* they take us out of the *<x,y>* plane. We can look at null geodesics, and shorten notation a bit by calling the first parenthesized term: $T \equiv 1-\frac{2\,m}{\sqrt{r^2-a^2}}$. Now when *ds=0,*

$$\sqrt{T}\left(dx^0 + \frac{dr}{r/2\,m-1}\right) = \pm\, dr \quad , \qquad \text{so:} \quad \frac{dr}{dx^0} = \frac{\sqrt{T}}{1-\dfrac{\sqrt{T}}{r/2\,m-1}} \quad . \tag{24}$$

We can see two terms of interest: T has a pole at *r=a,* and, distinctly, the denominator in the time transform becomes zero as usual for r nearing $m.$ We are not interested in this regime at the moment. For radii of $r>a$, T is close to unity, but as the critical radius a is approached, on a scale of *m,* it goes down through zero and blows up asymptotically as $r \twoheadrightarrow a$, *with a minus* sign. Regardless of the confusion of an imaginary root, at this pole the whole fraction becomes close to:

$$\frac{dr}{dx^0} \to -(r/2m-1) \quad , \tag{25}$$

a large quantity. This is strange since it is not matched by behavior approaching this point from the inside. Our suspicions of odd results surrounding the indefinite character of g_{00} at the critical radius are certainly born out.

Let us ask now about angular propagations in ϕ, which is to say differentials $d\hat{\phi}$. We expect complications as with radial propagations:

$$ds^2 = (1-2m/\rho)(d\hat{x}^0)^2 - r^2(1+2ma^2/\rho r^2)\, d\hat{\phi}^2 - 4\frac{am}{\rho}\, d\hat{x}^0\, d\hat{\phi} \quad . \tag{26}$$

Indeed the plot thickens with the need to include the cross-term. For the null form:

$$0 = -(1-2m/\rho) + r^2\left(1+2\frac{m}{\rho}\frac{a^2}{r^2}\right)\left(\frac{d\hat{\phi}}{d\hat{x}^0}\right)^2 + 4\frac{ma}{\rho}\frac{d\hat{\phi}}{d\hat{x}^0} \quad . \tag{27}$$

This is a quadratic form and we may solve it, defining *N=a/r:*

$$r\frac{d\hat{\phi}}{d\hat{x}^0} = \frac{1-2\,Nm/\rho}{1+2\,N^2 m/\rho} \quad . \tag{28}$$

This generalizes $(1-2\frac{m}{\rho})$ which was earlier called "*T*" and we can see behavior gets strong as ρ goes to zero. The fraction in the farfield becomes unity, as $N \to 0$.

It is more subtle to interpret a Cartesian differential in *z.* In the region near the *z=0* plane, we can now deal with: $\cos\hat{\theta} = z/\rho$. As long as we are outside the critical radius this is well defined. If ρ is finite, then it will not depend immediately on a change in z so we may say: $-\sin\hat{\theta}\, d\hat{\theta} = dz/\rho$. Since we assume we are near $\hat{\theta} = \pi/2$ we can ignore, to first order, changes in $\sin\hat{\theta}$ and write the metric term as:

$$-\rho^2\, d\hat{\theta}^2 = -\frac{dz^2}{\sin^2\hat{\theta}} \quad , \qquad \text{or simply} \quad -dz^2 \quad . \tag{29}$$

The metric form is: $$ds^2 = (1-2m/\rho)(d\,\hat{x}^0)^2 - dz^2 \quad , \tag{30}$$

which is quite like the radial solution for null paths: $$\pm dz = (1-2\,m/\rho)\,d\,\hat{x}^0 \quad . \tag{31}$$

This completes the study of null geodesics in the plane *z=0.* Behavior is asymptotic approaching the ring of *r=a* from outside, though the implied scale is $2m$. Since $a \gg 2m$ this invites physical interpretation. One might wonder about the choice of the (+) sign in the quadratic solution for ρ^2 but this is the only useful choice to meet a “flat” far-field constraint.

Part IV: The Fourth Dimension

The final simple analytic question is about the field in the locus *<x,y>*=0. As we go up on the *z*-axis, it is clear that *r=z=ρ* and the polar angle is zero, so: $\cos\hat{\theta}=1$. The metric form is:

$$ds^2 = \left(1-\frac{2\,m\,\rho}{\rho^2+a^2}\right)(d\,\hat{x}^0)^2 - \frac{\rho^2+a^2}{\rho^2+a^2-2\,m\,\rho}\,d\,\rho^2 - (\rho^2+a^2)\,d\,\hat{\theta}^2 \quad . \tag{32}$$

Since denominators contain a^2 the *m-* terms will be insignificantly small. Thus:

$$ds^2 = (d\,\hat{x}^0)^2 - d\,\rho^2 - (z^2+a^2)\,d\hat{\theta}^2 \quad . \tag{33}$$

The only job is to figure the space terms for deviations off-axis, so we can examine null geodesics of $dr_c = d\sqrt{(x^2+y^2)}$. It is fairly easy to convince ourselves that there is no first-order change from the *ρ*-term, since a small move in *x* or *y* has only second-order effect in a sum of squares. So such differentials should map to $d\hat{\theta}$. Starting with the definition: $\cos\hat{\theta}=z/\rho$, we may consider *z* constant and ask about changes in *ρ.* Here: $$\rho^2 = 1/2(z^2+r_c^2-a^2) + \sqrt{1/4(z^2+r_c^2-a^2)^2+a^2z^2} \quad . \tag{34}$$

Differentiating the cosine, we get: $$-\sin\hat{\theta}\,d\hat{\theta} = -\frac{z}{\rho^2}\frac{d\rho}{dr_c}dr_c \quad . \tag{35}$$

However one accomplishes this, a little algebra shows: $$\frac{d\rho}{dr_c} = \frac{r_c z}{a^2+z^2} \quad , \tag{36}$$

so we substitute and see: $$d\hat{\theta} = \frac{r_c z^2\,dr_c}{\rho^2 \sin\hat{\theta}\,(a^2+z^2)} \quad . \tag{37}$$

We see there are two very small quantities, so we seek a valid expression

for: $$\frac{r_c}{\sin\hat{\theta}} = \frac{r_c}{\sqrt{1-z^2/\rho^2}} \quad . \tag{38}$$

It is not difficult to show that: $$\frac{r_c}{\sin\hat{\theta}} = \sqrt{a^2+z^2} \quad , \tag{39}$$

so we can express the angular change: $$d\hat{\theta} = (a^2+z^2)^{-1/2}dr_c \tag{40}$$

where we allow z/ρ=1, to first order. Thus these dependencies fall away in the final expression: $$(z^2+a^2)(d\hat{\theta})^2 = dr_c^2 \quad . \tag{41}$$

This somewhat anticlimactic result shows that lightspeed as seen externally is not changed except close to the origin in *r,* in both the radial sense and the transverse sense, on the *z*-axis. If we think back to the massive case, here the lotus of the inner circles has opened up on the *z*-axis and folded outward into a small region in the *z=0* plane.

Bibliography

[1] Introduction to General Relativity, Adler, Bazin, Schiffer, McGraw-Hill, 1975, chapter 7.